The correspondence between Frege and Hilbert- commentary from the perspective of Sartre's philosophy

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Problem definition

The correspondence between Frege and Hilbert on fundamental questions of mathematics provides an interesting insight into the thinking of two leading mathematicians of the 20th century. This correspondence is primarily concerned with the meaning and function of the essential elements of mathematics. What is a definition? What is an axiom? What is the relationship between an axiom and a definition in mathematics? "What is the truth of the axioms based on?" is another question discussed in this exchange of ideas.

During the correspondence, it became apparent that the views of the two mathematicians on these questions differed considerably, so that Hilbert, probably also due to work overload, discontinued the discussion, although Frege would obviously have liked to continue it. Hilbert did not fulfil Frege's wish to publish the letters, but in some way, the exchange of ideas saw the light of day. (Felix Meiner Verlag, Gottlob Frege's correspondence, 1980)

Opinion is divided as to which of the two scholars is right or wrong. A commentator named Heinrich Scholz writes:

> *...today no one doubts that Frege, who himself created something so fundamentally new on the basis of the classical concept of science, was no longer able to grasp Hilbert's radical revolution of this concept of science, so that his critical remarks, which are in themselves quite astute and still worth reading today, must essentially be described as irrelevant. (S. 2)*

On the other hand, one can also find the following remark:

> *However, Frege's critical achievement vis-à-vis Hilbert's axiomatic method has increasingly been recognised and systematically appreciated in the relevant Frege research, e.g. by Dummett, Gabriel, von Kutschera and Resnik. (S. 2)*

Accordingly, scholars argue whether Hilbert or Frege should be given preference, which raises the question of what this dispute is based on. Frege and Hilbert are certainly very competent mathematicians. It cannot, therefore, be due to a lack of expertise. The divergent views of the commentators are certainly not due to a lack of insight either. It is much more likely that this dispute is based on a different *choice in* each case, a choice regarding the perspective one wishes to view mathematics, what is important to one in this complex phenomenon called 'mathematics' and what is not so important.

The concept of *choice* establishes the relationship between this dispute between Frege and Hilbert and Sartre's existentialism. Obviously, this basic concept of Sartre's philosophy, choice, also plays an important role in the sciences.

This also explains Hilbert's ambivalent attitude towards Frege. Hilbert is prepared to recognise the quality of Frege's considerations and even to concede that he needs to think about everything again in peace. Thus, Hilbert writes to Frege:

> *Unfortunately, due to the current overload of work of all kinds, I am unable to answer your letter in detail. Your comments are of great interest and value to me. They will certainly encourage me to think more carefully and formulate my thoughts more carefully. (S. 20)*

It is clear that Hilbert values Frege's thoughts and is far from simply considering these ideas to be wrong. Something else prevents him from adopting them. It is Hilbert's *view* of mathematics that makes him a sceptic about Frege's ideas.

In this essay, only one aspect of the discussion will be examined: The relationship between a definition and an axiom in the context of mathematics, especially geometry, but also in other fields.

Frege's position

Frege's criticism of Hilbert centres on the relationship between a definition and an axiom. He complains about Hilbert because he does not draw the dividing line between these two concepts enough, so the result is chaos and confusion:

> *However, people are also unclear about what they call a point. At first, they think of points in terms of Euclidean geometry, which is reinforced by the theorem that the axioms express basic facts of our perception. Later, however, you think of a pair of numbers as a point. I am concerned about the sentences that the exact and complete description of relationships is provided by the axioms of geometry and that axioms define the term "between". This imposes something on the axioms that is a matter of definition. It seems to me that this blurs the boundaries between definitions and axioms in an alarming way and that, in addition to the old meaning of the word "axiom",...another, but to me not quite comprehensible, meaning emerges. (S. 7/8)*

The difference between a definition and an axiom is important for Frege because a definition establishes a fact for him. This fact is then true *by definition*. Other concepts and their corresponding facts, such as axioms, principles and theorems, are not true by definition but for other reasons. An axiom of geometry, for example, is true according to Frege because its truth is grounded in observation. On the other hand, the truth of theorems must be demonstrated by evidence. Frege believes this is the correct structure of a mathematical theory, and he demands that mathematicians should adhere to it because otherwise, chaos and confusion threaten.

Hilbert's answer

Hilbert rejects Frege's criticism. However, his reasons for doing so are complex and cannot be reduced to the fact that he considers Frege's views unjustified. He seems to consider Frege's ideas correct but sees them as a one-sided and limited perspective on mathematics. Of course, Frege's concern for clarity and unambiguity of conceptualisations is to be supported, but not at any price, for example, not at the price of problem-solving competence. Hilbert writes to Frege:

> *If we want to understand each other, we must not forget the diversity of the intentions that guide us. I have been forced to set up my system of axioms by necessity: I wanted to give the possibility of understanding those geometrical propositions which I consider to be the most important results of geometrical research: that the axiom of parallels is not a consequence of the other axioms...I wanted to answer the question of whether the proposition that in two equal rectangles with the same base line the sides are also equal can be proved, or is rather a new postulate, as in Euclid. (S. 11)*

Hilbert brings to light an argument that Frege did not even have on his radar: the question of the *purpose* of mathematical research. At this point, two further concepts become visible that are important for Sartre's philosophy: the *design* and the *individual*. The mathematician's research is based on the internalisation of the objectively given and on an individual design of the goal to be achieved.

However, Hilbert also emphasises the success that his perspective has brought to mathematics:

> *That my system of axioms allows such questions to be answered in a very specific way and that one receives very surprising and even quite unexpected answers to many of these questions is, I believe, taught in my Festschrift...This, then, is my main intention. (S. 11)*

The following is recognisable: Hilbert believes that the axiomatisation of mathematics should not be dogmatic and dictatorial. Clarity, unambiguity and certainty are essential, but they are not the only criteria. Ultimately, the adequacy of an axiom system is reflected in its problem-solving competence, and it is up to the *freedom* of the researcher to determine the adequacy of the axiom system depending on the problem situation. Hilbert basically says to Frege: "Your logical efforts are honoured, but please do not take away my freedom to think and research in my way".

Hilbert also points out the difference between the creative freedom of the researcher and arbitrary freedom. Creative freedom consists of the researcher asking a skilful question based on the given and then trying to find an answer to this question. In Hilbert's case, the question is whether Euclid's postulate can be proved; he answers that he sets up a new system of axioms, the adequacy of which is based on his success with the problem posed. The truth of the axioms is ensured by their freedom from contradiction.

What is important, therefore, is the context of the *entire system* and not the individual elements of this system. How one names these individual elements, whether axiom or explanation, is of secondary importance to Hilbert:

> *If you would prefer to call my axioms characteristics of the terms placed in the "explanations" and thus existing, I would have no objection at all, except that this contradicts the habit of mathematicians and physicists. (S. 12)*

Hilbert, therefore, insists that the work of the creative mathematician corresponds to a kind of limbo between freedom and necessity. Necessities surround the mathematician, but he must ensure that he has enough freedom to deal with these necessities. The necessities must not become a straitjacket that deprives him of room for manoeuvre. In particular, conceptual fetishism should be avoided. The focus must always be on the actual problems, not the concepts:

> *My opinion is that a concept can only be logically defined by its relationships to other concepts. I call these relationships, formulated in certain statements, axioms and thus arrive at the conclusion that the axioms...are the definitions of the concepts. I did not come up with this view for the sake of amusement, but I saw myself compelled to do so by the demand for rigour in logical reasoning and in the logical construction of a theory. I have come to the conviction that in mathematics and the natural sciences one can only treat more subtle things with certainty in this way, otherwise one is merely going round in circles. (S. 23)*

So, suppose you want to evaluate an axiom system. In that case, it is not enough to ensure the exact differentiation of the individual elements, such as definition and axiom. Still, you have to keep an eye on the interaction of the entire system and judge it based on its problem-solving competence. Even if Frege succeeds in setting up a system of axioms in his sense, he will end up just going around in circles. Hilbert believes that only his method can turn the mathematical glass bead game into a fruitful scientific event that is also capable of solving more subtle problems.

In other words, Hilbert thinks that the foundations of mathematics are not set in stone but should be adapted to the respective mathematical epoch with its specific problems. This can even go so far that the meaning of the word 'axiom' must be reassessed. It would, therefore, be one-sided to see mathematics as a system that is put together like a tower of Lego bricks, with the axioms at the bottom and the theorems at the top. Instead, it is a cycle in which the theorems depend on the axioms, and the axioms depend on the theorems.

Modern mathematics confirms this point of view. The Hilbert space theory, for example, is a system of axioms in which the theorems can be deduced from these axioms. However, the axioms themselves are the result of mathematical research in which many of the theorems were known before the axioms. Consequently, it must be possible to adapt the axiom system to the theorems.

Commentary on the Frege versus Hilbert controversy from the perspective of Sartre's philosophy

Frege and Hilbert pursue different plans in their respective research, and Hilbert points out that his plan differs from Frege's. Therefore, it is understandable that Frege considers certain aspects particularly important, while Hilbert emphasises other aspects.

Hilbert clarifies that the term 'mathematics' must be placed in inverted commas. It does make sense to speak of 'mathematics' because mathematics exists as an objective fact. It exists, for example, as a school subject or a department of academic faculties. There are maths teachers and maths professors. It also makes sense to describe a 'mathematician' as a job title.

However, this does not change the fact that the *researching* mathematician must internalise the objectively given external fact of 'mathematics', that he must give this external fact a subjective colouring about the foreground-background structuring so that he as an *individual* can deal creatively with the facts.

In other words, the *creative* mathematician is forced to critically engage with the given and scrutinise traditional concepts regarding their problem-solving competence. In this process, it must be possible to change given schemes, for example, to adapt the meaning of the word 'axiom' to the problem at hand.

This brings to light another concept of Sartre's philosophy: freedom. *Man is condemned to freedom, which* is a famous slogan of existentialism. For Sartre, man is freedom. Consequently, freedom must be relevant to *all* aspects of being-in-the-world, simply by virtue of the fact that it is *human beings* who experience this being-in-the-world. In other words, even the researcher should not overlook the structures of human existence.

One mathematician particularly emphasised the importance of freedom for mathematics: Georg Cantor. In Sartre's sense, it is interesting in this context that Cantor defended his transfinite numbers with the remark that the real essence of mathematics lies in freedom. Opponents of Cantor, for example, his teacher Kronecker, countered with the remark that the essence of mathematics does not lie in freedom, but in truth. In Sartre's sense, one would have to say:

> *The basis of truth is freedom. (Sartre, Truth and Existence)*

Hilbert argued sharply against Kronecker in the Cantor versus Kronecker dispute:

> *Kronecker coined the motto: God created the whole number, everything else is the work of man. Accordingly, he - the classic prohibitionist dictator - frowned upon anything that was not an integer for him, but on the other hand, he and his school were not interested in thinking any further about the integer itself. (Büttemeyer, Philosophy of Mathematics, Karl Alber, p. 116)*

It is easy to recognise that the Frege versus Hilbert dispute mirrors the dispute between Cantor and Kronecker. In particular, the expression 'prohibition dictator' applies to Kronecker and Frege.

The question of the 'essence of maths' is fundamental. Is certainty and clarity of the fundamentals of mathematics paramount, or should one pursue a pragmatic path to achieve maximum problem-solving competence, even at the cost of a fluctuating foundation of mathematics?

Of course, it is not a question of doubting the value of a secure foundation in maths but of positioning this value to other values, such as problem-solving skills. The corresponding decision must be made according to the situation and can vary from era to era.

Therefore, it is possible that in a particular epoch, problem-solving skills take centre stage, while in another epoch, the question dominates what the basis of mathematics is. At the time of the great creative mathematicians—Newton, Leibniz, Euler, and Cauchy—it was more a question of developing new concepts on a fluctuating basis. The development of infinitesimal calculus initially demanded that the logical foundations of mathematics not be taken too seriously. This type of mathematics was heavily criticised by contemporaries, such as Berkeley, for precisely this reason.

In the era of the Enlightenment - Weierstrass, Dedekind - the focus was more on clarifying the basic concepts of mathematics by attempting to support the fluctuating basis of analysis and number theory with a firm logical foundation. But even in this era, there were disputes about the priority of certainty and freedom within mathematics.

Which point of view is favoured in a particular epoch can only be decided with the help of a life plan. For it is impossible to hope to transform oneself as a human being into an objective eye of the world in order to then determine *for eternity* what is right and important in mathematics from a god-like position through insight into divine truth. Rather, what is needed is a life plan that must bring about the decision.

Mathematics: concept or notion?

This is probably the weakness of Frege's point of view: He wants to cement the basis of mathematics for all mathematicians and eternity, and this is precisely what is impossible, namely because of the structures of human existence. Frege seems to want to see an unassailable fortress in the logical foundation of mathematics along the lines of Martin Luther:

A strong fortress is our God,

a good defence and weapons.

About geometry, he sees the basis of this solid castle in the visualisation and correct application of the elementary concepts of mathematics: axiom, definition, explanation, and theorem. Concerning number theory, he considers his concept of the *scope of concepts* to be authoritative. Hilbert is opposed to this view:

> *Frege attempted to base the theory of numbers on pure logic, Dedekind on set theory as a chapter of pure logic: neither of them achieved their goal. Frege did not handle the usual conceptualisations of logic carefully enough in their application to mathematics: he considered the scope of a concept to be something given without further ado in such a way that he then believed he could take these scopes as things themselves without restriction. He thus to a certain extent fell into an extreme conceptual realism. (Büttemeyer, p. 137)*

It is one of the oddities of the history of mathematics that Frege, the great logician who attached so much importance to the exactness of conceptualisations, constructed a self-contradictory system. Bertrand Russell proved this with the antinomy of the set of all sets that do not contain themselves as an element, which he discovered. Hilbert's text above is an allusion to Frege's misadventure.

Hilbert, on the other hand, seems to want to say that the foundations of mathematics must adapt to the respective problem situations, whereby extreme conceptual realism should be avoided. In other words, the meaning of the words 'axiom' and 'definition' is a significant problem, but this problem should always be solved depending on the particular problem situation. For Hilbert, mathematical *practice is* always in the foreground, and the conceptualisations are to be assessed concerning this practice.

Basically, everything must be put to the test: Even the terms "geometry", "mathematics" and "science" are subject to change over time. For example, today, we speak of "physical mathematics", whereas "mathematical physics" has been used for some time. The perfection of electronic computers could also lead to a change in the concept of mathematics.

In the context of Sartre's philosophy, this difference between Frege and Hilbert can be expressed by the terms 'concept' and 'notion'. In Sartre's language, mathematics in Hilbert's sense would be a 'notion', not a 'concept'. The following explanation can be found in a lexicon:

> *Sartre distinguishes concept and notion and identifies the latter with a thinking in motion that gradually gives itself its determinations by endeavouring to reach the concrete. (Dictionnaire Sartre, translation into German: Alfred Dandyk)*

This formulation illustrates the difference between Frege and Hilbert. The actual goal of mathematics is the concrete. What the concrete should be, however, must be decided anew by everyone in each epoch. For Frege, it was the logical foundation of mathematics; for Hilbert, it was the further development of mathematics in areas relevant to practice, for example, the theory of partial differential equations. According to this life plan, the basic concepts of mathematics may take on a new meaning so that mathematics, viewed historically, presents itself as thinking in motion.

It seems that Frege rejects this historical view. He is not looking for thinking in motion but a Platonic concept of mathematics with eternal value. The matter is complicated by the fact that Frege is not entirely wrong. Mathematics plays a unique role in the history of science: it is mainly an earthly image of the Platonic heaven of ideas. The only question is whether it can be a perfect and complete image or whether, in a certain sense, it is only an approximation of knowledge.

Sartre writes the following on the problem of the historicity of mathematics:

> *Euclidean geometry, Cartesian analytics and Newtonian physics, for example, are true. But their relationships to later truths are different. The relation of Euclidean geometry to non-Euclidean geometries, for example, is an exteriority relation...Newtonian physics, on the other hand, is integrated into modern physics, which, without negating it, gives it an inner limitation: it becomes the physics of appearances, the physics of the as if, the physics of the special case. (Sartre, Truth and Existence)*

Here, Sartre combines mathematical Platonism with the historicity of mathematics. In other words, for him, the Platonic truth of mathematics and its historicity are not contradictory. Euclidean geometry, for example, is *true,* and in this respect, it is a vivid example of a Platonic system of ideas. There is no doubt that Euclid's work must be seen in the context of the philosophy of Platonism. In this respect, Frege's view is understandable.

On the other hand, today, we see Euclidean geometry with different eyes than in Euclid's time. We know, for example, that there are also non-Euclidean geometries. We also know that geometries of different dimensions can be constructed, such as four-dimensional geometries, as used in the theory of relativity.

This means that Euclidean geometry has lost its claim to absoluteness; it is one geometry among others, and this also changes the idea of in which sense Euclidean geometry is true and in which sense it is not. Pythagoras' theorem, for example, is only valid under certain conditions and not in every geometry.

Although nothing has changed within Euclidean geometry, it has undergone a comprehensive transformation regarding its external relationships. It competes with other geometries and has only a limited field of application. In one sense, it has remained the same theory; in another sense, it is now a new theory.

The mathematicians of each epoch, therefore, have the task of integrating the old knowledge, which in a certain sense has eternal value, into the totality of contemporary knowledge and thus historicising it. The mathematician is, therefore, in reality, dealing with the incarnation of the eternal in the temporal.

The weakness of Frege's view seems to lie in the fact that he wants to declare this hybrid nature of mathematics, the representation of the eternal in the temporal, to be the prehistory of mathematics. He sees his task as ending this prehistory and explaining to mankind once and for all what is meant by the word 'axiom' and what a definition is. Hilbert opposes Frege's dictatorship. What Hegel claimed for philosophy, namely the end of philosophy, Frege seems to claim for mathematics. An end to this chaos and confusion in mathematics. Now, the terms are defined, and that's where it should stay. That's it!